Minimality and Maximality of Ordered Quasi-Ternary Γ-Ideals in Ordered Ternary Γ-Semirings

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Abstract — **Our aim in this paper is to develop a body of results on the minimality and maximality of ordered quasi-Γ-ideals in ordered ternary Γ-semirings, that can be used like the more classical results on unordered structures.**

Index Terms — **ordered ternary Γ-semiring, ordered quasi-Γ-ideal, minimality and maximality.**

I. Introduction

The literature of ternary algebraic system was introduced by Lehmer [18] in 1932. He investigated certain ternary algebraic systems called triplexes which turn out to be ternary groups. The notion of ternary semigroups was known to Banach. He showed by an example that a ternary semigroup does not necessarily reduce to an ordinary semigroup. We can see that any semigroup can be reduced to a ternary semigroup. The study of ordered ternary semigroups began about 2000 by several authors, for example, Iampan [15], Chinram [8], and Akram and Yaqoob [1]. The theory of different types of ideals in (ordered) semigroups and in (ordered) ternary semigroups was studied by several researches such as: In 1965, Sioson [25] studied ideal theory in ternary semigroups. He also introduced the notion of regular ternary semigroups and characterized them by using the notion of quasi-ideals. In 1995, Dixit and Dewan [11] studied the properties of quasi-ideals and bi-ideals in ternary semigroups. In 1998, the concept and notion of ordered quasi-ideals in ordered semigroups was introduced by Kehayopulu [17]. In 2000, Cao and Xu [4] characterized minimal and maximal left ideals in ordered semigroups, and gave some characterizations of minimal and maximal left ideals in ordered semigroups. In 2002, Arslanov and Kehayopulu [2] gave some characterizations of minimal and maximal ideals in ordered semigroups. In 2004, Iampan and Siripitukdet [16] characterized (0-)minimal and maximal ordered left ideals in ordered Γ-semigroups, and gave some characterizations of (0-)minimal and maximal ordered left ideals in ordered Γ-semigroups. In 2007, Iampan [13] characterized (0-)minimal and maximal lateral ideals in ternary semigroups. In 2008, Iampan [14] characterized (0-)minimal and maximal ordered quasi-ideals in ordered semigroups, and gave some characterizations of (0-)minimal and maximal ordered quasiideals in ordered semigroups. Dutta, Kar and Maity [12] studied some interesting properties of regular ternary semigroups, completely regular ternary semigroups, intra-regular ternary semigroups and characterized them by using various ideals of ternary semigroups. In 2009, Bashir and Shabir [3] introduced the notions of pure ideals, weakly pure ideals in ternary semigroups. They also defined purely prime ideals of a ternary semigroup and studied some properties of these ideals. In 2010, Iampan [15] introduced the concept of ordered ideal extensions in ordered ternary semigroups. In 2011, Saelee and Chinram [21] studied rough, fuzzy and rough fuzzy bi-ideals in ternary semigroups. In 2012, Changphas [5] studied minimal quasi-ideals in ternary semigroups. Choosuwan and Chinram [9] gave some characterizations of minimal and maximal quasi-ideals in ternary semigroups. Chinram, Baupradist and Saelee [7] characterized minimal and maximal bi-ideals in ordered ternary semigroups. Daddi and Pawar introduced the concepts of ordered quasi-ideals, ordered bi-ideals in ordered ternary semigroups, and studied their properties. Lekkoksung and Lekkoksung [19] gave some characterizations of intra-regular ordered ternary semigroups in terms of bi-ideals and quasi-ideals, bi-ideals and left ideals, and bi-ideals and right ideals in ordered ternary semigroups. Changphas [6] studied the properties of quasi-ideals and bi-ideals in ordered ternary semigroups. In 2013, Sanborisoot and Changphas [22] introduced the concepts of pure ideals, weakly pure ideals and purely prime ideals in ordered ternary semigroups. The notion of quasi-ideals in semigroups was first introduced by Steinfeld [25] in 1956, and it has been widely studied. In 1956, Steinfeld [26] gave some characterizations of 0-minimal quasi-ideals in semigroups. The concept of a (0-)minimal and a maximal one-sided ideal or ideal is the really interested and important thing in the many algebraic structures. The main purpose of this paper is to develop a body of results on the minimality and maximality of ordered quasi-ideals in ordered ternary semigroups, that can be used like the more classical results on unordered structures which studied by Choosuwan and Chinram [9]. In 2015 Sajani Lavanya, Madhusudhana Rao and Syam Julius Rajendra [23] introduced the notion of quasi ternary Γ-ideals and bi-ternary Γ-ideals in ternary Γ-semirings.

Before going to prove the main results we need the following definitions that we use later.

Definition I.1[27]: Let T and Γ be two additive commutative semigroups. T is said to be a ***Ternary Γ-semiring*** if there exist a mapping from T ×Γ× T ×Γ× T to T which maps $(x_1, \alpha, x_2, \beta, x_3) \rightarrow [x_1\alpha x_2\beta x_3]$ satisfying the conditions :
i) $[[a\alpha b\beta c]\gamma d\delta e] = [a\alpha[b\beta c\gamma d]\delta e] = [a\alpha b\beta[c\gamma d\delta e]]$
ii) $[(a + b)\alpha c\beta d] = [a\alpha c\beta d] + [b\alpha c\beta d]$
iii) $[a\alpha (b + c)\beta d] = [a\alpha b\beta d] + [a\alpha c\beta d]$
iv) $[a\alpha b\beta(c + d)] = [a\alpha b\beta c] + [a\alpha b\beta d]$ for all $a, b, c, d\in$ T and $\alpha, \beta, \gamma, \delta \in \Gamma$.

Obviously, every ternary semiring T is a ternary Γ-semiring. Let T be a ternary semiring and Γ be a commutative ternary semigroup. Define a mapping T ×Γ× T ×Γ× T ⟶ T by $a\alpha b\beta c = abc$ for all $a, b, c \in$ T and $\alpha, \beta \in \Gamma$. Then T is a ternary Γ-semiring.

Note I.2[27] : Let (T,Γ, +, [ ]) be a ternary Γ-semiring. For nonempty subsets A_1,A_2 and A_3 of T, let [AΓBΓC]= $\{\Sigma a\alpha b\beta c : a \in A, b \in B, c \in C, \alpha, \beta \in \Gamma\}$ .For $x\in$ T, let $[x\Gamma A_1\Gamma A_2] = [\{x\}\Gamma A_1\Gamma A_2]$. The other cases can be defined analogously.

Note I.3 [**27**]: Let T be a ternary semiring. If A, B are two subsets of T , we shall denote the set A + B = $\{a + b : a \in A, b \in B\}$ and 2A = { $a + a$: $a\in$ A}.

Definition I.4 [27]:A ternary Γ-semiring T is called an ***ordered ternary Γ-semiring*** if there is a partial order ≤ on T such that x ≤ y implies that (i) $a + c \le b + c$ and $c + a \le c + b$ (ii) $[a\alpha c\beta d] \le [b\alpha c\beta d]$, $[c\alpha a\beta d] \le [c\alpha b\beta d]$ and $[c\alpha d\beta a] \le [c\alpha d\beta b]$for all $a, b, c, d\in$ T and $\alpha, \beta, \gamma, \delta \in \Gamma$.

Note I.5 [**27**]: For the convenience we write $x_1\alpha x_2\beta x_3$ instead of $[x_1\alpha x_2\beta x_3]$

Notation I.6 [27]: Let T be PO-ternary Γ-semiring and S be a nonempty subset of T. If H is a nonempty subset of S, we denote {s∈S :$s \le h$ for some $h\in$ H} by $(H]_S$.

Notation I.7 [27]: Let T be PO-ternary Γ-semiring and S be a nonempty subset of T. If H is a nonempty subset of S, we denote {s∈S :$h \le s$ for some $h\in$ H} by $[H)_S$.

Note I.8 [27]: $(H]_T$ and $[H)_T$ are simply denoted by (H] and [H) respectively.

Definition I.9 [27]: Let T be PO-ternary Γ-semiring. A nonempty subset 'S' is said to be a PO-***ternary Γ-sub semi ring*** of T if

(i) S is an additive sub semi group of T,
(ii) $a\alpha b\beta c\in$ S for all $a, b, c\in$ S, $\alpha, \beta \in \Gamma$.
(iii) $T\in$ T, s∈ S, $t \le s \Rightarrow t\in$ S.

Note I.10 [27]: A nonempty subset S of a po-ternary Γ-semiring T is a po-ternary Γ-sub semi ring of T iff (1)S + S ⊆ S (2) SΓSΓS ⊆ S, (2) (S] ⊆ S.

Theorem I.11 [27]: Let S be po-ternary Γ-semi ring and A ⊆ S, B ⊆ S. Then (i) A ⊆ (A], (ii) ((A]] = (A], (iii) (A]Γ(B]Γ(C] ⊆ (AΓBΓC] and (iv) A ⊆ B ⇒ A ⊆ (B], (v) A ⊆ B ⇒(A] ⊆ (B], (vi) (A ∩ B] = (A] ∩ (B], (vii) (A ∪ B] = (A] ∪ (B].

Definition I.12: An element z of an ordered ternary **Γ**-semiring T is said to be a ***zero element*** if (1) $z\alpha x\beta y = x\alpha z\beta y = x\alpha y\beta z = z$ for all $x, y \in$ T, $\alpha, \beta \in \Gamma$ and (2) $z \le x$ for all $x \in$ T. If $z \in$ T is a zero element, it is denoted by 0.

Definition I.13 [27]: A nonempty subset A of a PO-ternary Γ-semiring T is said to be ***left PO-ternary Γ-ideal*** of T if
(1) $a, b \in$ A implies $a + b\in$ A.
(2) $b, c\in$ T, $a\in$ A, $\alpha, \beta\in \Gamma$ implies $b\alpha c\beta a\in$ A.
(3) $t \in T, a \in A$, $t \le a \Rightarrow t\in$ A.

Note I.14 [27]: A nonempty subset A of a PO-ternary Γ-semiring T is a left PO-ternary Γ-ideal of T if and only if A is additive subsemigroup of T, TΓTΓA ⊆ A and (A] ⊆ A.

Definition I.15 [27] : A nonempty subset of a PO-ternary Γ-semiring T is said to be a ***lateral PO-ternary ideal*** of T if
(1) $a, b \in$ A implies $a + b\in$ A.
(2) $b, c\in$ T , $\alpha, \beta\in \Gamma$, $a\in$ A implies $b\alpha a\beta c\in$ A.
(3) $t \in T, a \in A$, $t \le a \Rightarrow t \in$ A.

Note I.16 [27]: A nonempty subset of A of a PO-ternary semiring T is a lateral PO-ternary Γ-ideal of T if and only if A is additive sub semi group of T, TΓAΓT ⊆ A and (A] ⊆ A.

Definition I.17 [27] : A nonempty subset A of a PO-ternary Γ-semiring T is a ***right PO-ternary Γ-ideal*** of T if
(1) $a, b \in$ A implies $a + b\in$ A.
(2) $b, c \in$ T , $\alpha, \beta \in \Gamma$, $a\in$ A implies $a\alpha b\beta c\in$ A.
(3) $t \in T, a \in A$, $t \le a \Rightarrow t\in$ A.

Note I.18 [27] : A nonempty subset A of a PO-ternary Γ-semiring T is a right PO-ternary Γ-ideal of T if and only if A is additive sub semi group of T, AΓTΓT ⊆ A and (A] ⊆ A.

Definition I.19 [27]: A nonempty subset A of a PO-ternary Γ-semiring T is a ***two sided PO-ternary Γ-ideal*** of T if

(1) $a, b \in$ A implies $a + b\in$ A
(2) $b, c\in$ T ,$\alpha, \beta \in \Gamma$, $a\in$A implies $b\alpha c\beta a\in$ A, $a\alpha b\beta c\in$ A.
(3) $t \in T, a \in A$, $t \le a \Rightarrow t \in$ A.

Note I.20 [27] : A nonempty subset A of a PO-ternary Γ-semiring T is a two sided PO-ternary Γ-ideal of T if and only if it is both a left PO-ternary Γ-ideal and a right PO-ternary Γ-ideal of T.

Definition I.21 [27] : A nonempty subset A of a PO-ternary Γ-semiring T is said to be ***PO-ternary Γ-ideal*** of T if
(1) $a, b \in$ A implies $a + b\in$ A
(2) $b, c \in$ T ,$\alpha, \beta \in \Gamma$, $a\in$ A implies $b\alpha c\beta a\in$ A, $b\alpha a\beta c\in$ A, $a\alpha b\beta c\in$ A.
(3) $t \in T, a \in A$, $t \le a \Rightarrow t \in$ A.

Note I.22 [27] : A nonempty subset A of a PO-ternary Γ-semiring T is aPO-ternary Γ-ideal of T if and only if it is left PO-ternary Γ-ideal, lateral PO-ternary Γ-ideal and right PO-ternary Γ-ideal of T.

II. Ordered Quasi-ternary Γ-Ideals and Ordered Bi-ternary Γ-Ideals

Definition II.1: An additive sub semi group Q of an ordered ternary Γ-semiring T is said to be an ***ordered quasi-Ternary Γ-ideal*** of T if
(1) (TΓTΓQ]∩(TΓQΓT]∩(QΓTΓT] ⊆ Q,
(2) (TΓTΓQ]∩(TΓTΓQΓTΓT]∩(QΓTΓT] ⊆ Q, and
(3) (Q] ⊆ Q.

Note II.2: We can easily prove that {0} is the smallest ordered quasi-Ternary Γ-ideal of an ordered ternary Γ-semiring T with a zero element and it is called a zero ordered quasi-Ternary Γ-ideal of T. Moreover, 0 ∈ Q for all ordered quasi-Ternary Γ-ideal Q of T.

Definition II.3: An ordered ternary Γ-sub semi ring B of an ordered ternary Γ-semiring T is said to be an ***ordered bi-Ternary Γ-ideal*** of T if
(1) BΓTΓBΓTΓB ⊆ B, and
(2) (B] ⊆ B.

Lemma II.4: Let T be an ordered ternary Γ-semi ring. Then the following statements hold.
(1) Every ordered left, ordered lateral and ordered right ternary Γ-ideal of T is an ordered quasi-ternary Γ-ideal of T.
(2) The intersection of an ordered left, an ordered lateral and an ordered right ternary Γ-ideal of T is an ordered quasi-ternary Γ-ideal of T.
(3) Every ordered quasi-ternary Γ-ideal of T is an ordered bi-ternary Γ-ideal of T.

Proof: **(1)** Let L, R and M be an ordered left, an ordered right and an ordered lateral Ternary Γ-ideal of T, respectively. (1) We see that (L] = L, (R] = R and (M] = M.
Thus (TΓTΓL] ∩ (TΓLΓT ∪ TΓTΓLΓTΓT]∩(LΓTΓT]
⊆ (TΓTΓL] ⊆ (L] = L,
(TΓTΓR]∩(TΓRΓT ∪TΓTΓRΓTΓT]∩(RΓTΓT]
⊆ (RΓTΓT] ⊆ (R] = R, and
(TΓTΓM] ∩ (TΓMΓT ∪ TΓTΓMΓTΓT] ∩ (TΓTΓM]
⊆ (TΓMΓT ∪ TΓ(TΓMΓT)ΓT] ⊆ (M ∪TΓMΓT]
⊆ (M ∪M] = (M] = M.
Hence, L,R and M are ordered quasi-ternary Γ-ideals of T.
(2) Suppose that Q = L∩M ∩R and let l ∈ L, m ∈ M and r ∈ R, α, β ∈ Γ.
Then $r\alpha m\beta l$ ∈ RML ⊆ TΓTΓL∩TΓMΓT ∩RΓTΓT
⊆ L∩M∩R = Q, so Q ≠ ∅. We see that
(Q] = (L∩M∩R] ⊆ (L]∩(M]∩(R] = L∩M∩R = Q.
Thus (TΓTΓQ]∩(TΓQΓT ∪TΓTΓQΓTΓT]∩(TΓTΓQ]
⊆ (TΓTΓL]∩(TΓMΓT ∪TΓTΓMΓTΓT]∩(RΓTΓT]
⊆ (L]∩(M]∩(R] = L∩M ∩R = Q.
Hence, Q is an ordered quasi-Ternary Γ-ideal of T.

(3) Let B be an ordered quasi-Ternary Γ-ideal of T. Then BΓTΓBΓTΓB ⊆ (TΓTΓT)ΓTΓB ⊆ TΓTΓB, BΓTΓBΓTΓB ⊆ TΓTΓBΓTΓT ⊆ TΓBΓT ∪TΓTΓBΓTΓT and BΓTΓBΓTΓB ⊆ BΓTΓ(TΓTΓT) ⊆ BΓTΓT. Since B is an ordered quasi-Ternary Γ-ideal of T, we have BΓTΓBΓTΓB ⊆ TΓTΓB∩(TΓBΓT ∪TΓTΓBΓTΓT)∩BΓTΓT ⊆ (TΓTΓB]∩(TΓBΓT ∪TΓTΓBΓTΓT]∩(BΓTΓT] ⊆ B and (B] = B. Hence, B is an ordered bi-ternary Γ-ideal of T.

Theorem II.5: Let A be a nonempty subset of an ordered ternary Γ-semiring T. Then the following statements hold.
(1) (TΓTΓA], (AΓTΓT] and (TΓAΓT∪TΓTΓAΓTΓT] are an ordered left, an ordered right and an ordered lateral ternary Γ-ideals of T, respectively.
(2) (TTA∪A], (ATT ∪A] and (TAT ∪TTATT ∪A] are an ordered left, an ordered right and an ordered lateral ternary Γ-ideals of T containing A, respectively.

Proof: (1) Suppose that s, t ∈ (TΓTΓA], then there exist x_i, y_i, x_j, y_j ∈ T, α_i, β_i ∈ Γ and a ∈ A such that $s \le \sum_{i=1}^{n} x_i\alpha_i y_i\beta_i a$ and $t \le \sum_{j=1}^{n} x_j\alpha_j y_j\beta_j a$. Since T is a PO-ternary Γ-semiring and TΓTΓA is a left PO-ternary Ternary Γ-ideal of T.

We have $s + t \le \sum_{i=1}^{n} x_i\alpha_i y_i\beta_i a + \sum_{j=1}^{n} x_j\alpha_j y_j\beta_j a$ ∈TTA and hence $s + t$ ∈(TΓTΓA]. Similarly we can show that $s + t$ ∈ (TΓAΓT ∪ TΓTΓAΓTΓT] and $s + t$ ∈ (AΓTΓT].
Therefore (TΓTΓ*A*], (TΓAΓT∪TΓTΓAΓTΓT] and (AΓTΓT] are additive sub semi groups of T.
Since A ≠ ∅, we have (TΓTΓA] ≠ ∅, (AΓTΓT] ≠ ∅ and (TΓAΓT∪TΓTΓAΓTΓT] ≠ ∅.
We see that ((TΓTΓA]] = (TΓTΓA], ((AΓTΓT]] = (AΓTΓT] and ((TΓAΓT∪TΓTΓAΓTΓT]] = (TΓAΓT∪TΓTΓAΓTΓT].
Thus TΓTΓ(TΓTΓA] = (T]Γ(T]Γ(TΓTΓA] ⊆ ((TΓTΓT)ΓTΓA] ⊆ (TΓTΓA],
(AΓTΓT]ΓTΓT = (AΓTΓT]Γ(T]Γ(T]
⊆ (AΓTΓ(TΓTΓT)] ⊆ (AΓTΓT] and
TΓ(TΓAΓT ∪TΓTΓAΓTΓT]ΓT
= (T]Γ(TΓAΓT ∪TΓTΓAΓTΓT]Γ(T]
⊆ (TΓ(TΓAΓT ∪TΓTΓAΓTΓT)ΓT]
⊆ (TΓ(TΓAΓT)ΓT ∪TΓ(TΓTΓAΓTΓT)ΓT]
= ((TΓTΓT)ΓAΓ(TΓTΓT)∪TΓTΓAΓTΓT]
⊆ (TΓAΓT ∪TΓTΓAΓTΓT].
Hence, (TΓTΓA], (AΓTΓT] and (TΓAΓT ∪ TΓTΓAΓTΓT] are an ordered left, an ordered right and an ordered lateral ternary Γ-ideals of T, respectively.
(2) The proof is almost similar to the proof of (1).

Theorem II.6: If Q is an ordered quasi-Γ-ideal of an ordered ternary Γ-semiring T, then it is the intersection of an ordered left, an ordered right and an ordered lateral ternary Γ-ideal of T.

***Proof*:** Assume that Q is an ordered quasi-Ternary Γ-ideal of T and let L = (TΓTΓQ ∪ Q], R = (QΓTΓT ∪Q] and M = (TΓQΓT ∪TΓTΓQΓTΓT ∪Q]. By Theorem II.5(2), we have L, R and M are an ordered left, an ordered right and an ordered lateral ternary Γ-ideals of T containing Q, respectively. Thus Q ⊆ L∩M ∩R. Since Q is an ordered quasi-Ternary Γ-ideal of T, we have
L∩M ∩R = (TΓTΓQ∪Q]∩(TΓQΓT ∪TΓTΓQΓTΓT ∪Q]∩(QΓTΓT ∪Q]
=((TΓTΓQ]∩(TΓQΓT∪TΓTΓQΓTΓT] ∩(QΓTΓT])∪(Q] ⊆ Q∪(Q] = Q.
Hence, Q = L∩M ∩R, so Q is the intersection of an ordered left, an ordered right and an ordered lateral Ternary Γ-ideal of T.

Theorem II.7: Let T be an ordered ternary Γ-semiring. Then the intersection of arbitrary nonempty family of ordered quasi-ternary Γ-ideals of T is either empty or an ordered quasi-ternary Γ-ideal of T.
Proof. Let {Q_i | i ∈ I} be a nonempty family of ordered quasi-Γ-ideals of T and let Q = $\bigcap_{i\in I} Q_i \neq \varnothing$. We claim that Q is an ordered quasi-Ternary Γ-ideal of T. Since Q_i is an ordered quasi-Ternary Γ-ideal of T for all i ∈ I, we have

(TΓTΓQ]∩(TΓQΓTUTΓTΓQΓTΓT]∩ (QΓTΓT]
⊆ (TΓTΓQ_i]∩(TΓQ_iΓTUTΓTΓQ_iΓTΓT]∩(Q_iΓTΓT]
⊆ Q_i for all i ∈ I.

Thus (TΓTΓQ]∩(TΓQΓTUTΓTΓQΓTΓT]∩(QΓTΓT]

$$\subseteq \bigcap_{i\in I} Q_i = Q \text{ and}$$

$$(Q] = (\bigcap_{i\in I} Q_i\,] \subseteq \bigcap_{i\in I}(Q_i] = \bigcap_{i\in I} Q_i = Q.$$

Hence, Q is an ordered quasi-Ternary Γ-ideal of T.

Definition II.8: Let A be an additive sub semi group of an ordered ternary Γ-semiring T. The intersection of all ordered quasi-ternary Γ-ideals of T containing A is called the ***ordered quasi-Ternary Γ-ideal*** of T generated by A and is denoted by Q(A). Moreover, Q(A) is the smallest ordered quasi-Ternary Γ-ideal of T containing A. If A = {a}, we also write Q({a}) as Q(a).

Theorem II.9: Let A be an additive sub semigroup of an ordered ternary Γ-semiring T. Then Q(A) = (A]∪((TΓTΓA]∩(TΓAΓT∪TΓTΓAΓTΓT]∩(AΓTΓT]). In particular,

Q(a) = (a]∪((TΓTΓa]∩(TΓaΓT ∪TΓTΓaΓTΓT]∩(aΓTΓT]) for all a ∈ T.

Proof: By Theorem II.5 (2), we have (A∪TΓTΓA], (A∪AΓTΓT] and (A∪TΓAΓT ∪ TΓTΓAΓTΓT] are an ordered left, an ordered right and an ordered lateral ternary Γ-ideals of T containing A, respectively.
By Lemma II.4 (2), we have (A∪TΓTΓA]∩(A∪ TΓAΓT∪TΓTΓAΓTΓT]∩(A∪AΓTΓT] is an ordered quasi-Ternary Γ-ideal of T containing A. Thus
Q(A)⊆(A∪TΓTΓA]∩(A∪TΓAΓT TΓTΓAΓTΓT]∩(A∪AΓTΓT]
=(A]∪((TΓTΓA]∩(TΓAΓT ∪TΓTΓAΓTΓT]∩(AΓTΓT]).
By the proof of Theorem II.6, we have
(A]∪((TΓTΓA]∩(TΓAΓT ∪TΓTΓAΓTΓT]∩(AΓTΓT])
=(A∪TΓTΓA]∩(A∪TΓAΓT ∪TΓTΓAΓTΓT]∩(A∪AΓTΓT]
⊆(Q(A)∪TΓTΓ(Q(A))]∩(Q(A)∪TΓ(Q(A))ΓT
∪TΓTΓ(Q(A))ΓTΓT]∩(Q(A)∪(Q(A))ΓTΓT] ⊆ Q(A).
Hence, Q(A) = (A]∪((TΓTΓA]∩(TΓAΓT ∪TΓTΓAΓTΓT]∩ (AΓTΓT]).

III. Minimality Of Ordered Quasi-Ternary Γ-Ideals In Ordered Ternary Γ-Semirings

In this section, we characterize the relationship between the minimality of ordered quasi-ternary Γ-ideals and a quasi-simple and a 0-quasi-simple ordered ternary Γ-semirings.

Definition III.1: Let T be an ordered ternary Γ-semiring without a zero element. Then T is said to be ***quasi-simple*** if T has no proper ordered quasi-ternary Γ-ideals.

Theorem III.2: Let T be an ordered ternary Γ-semiring without a zero element. Then the following statements are equivalent.
(1) T is quasi-simple.
(2) (TΓTΓa]∩(TΓaΓT ∪TΓTΓaΓTΓT]∩(aΓTΓT] = T for all a ∈ T.
(3) Q(a) = T for all a ∈ T.

Proof: (1)⇒(2) Assume that T is quasi-simple and let a ∈ T. By Theorem II.5 (1), we have (TΓTΓa], (aΓTΓT] and (TΓaΓT∪TΓTΓaΓTΓT] are an ordered left, an ordered right and an ordered lateral ternary Γ-ideals of T, respectively. By Lemma II.4 (2), we have (TΓTΓa]∩(TΓaΓT ∪TΓTΓaΓTΓT]∩(aΓTΓT] is an ordered quasi-Ternary Γ-ideal of T. Since T is quasi-simple, we have (TΓTΓa]∩(TΓaΓT ∪TΓTΓaΓTΓT]∩(aΓTΓT] = T.

(2)⇒(3) Assume that (TΓTΓa]∩ (TΓaΓT∪TΓTΓaΓTΓT]∩ (aΓTΓT] = T for all a ∈ T. Let a ∈ T. Then (TΓTΓa]∩(TΓaΓT ∪TΓTΓaΓTΓT]∩(aΓTΓT] = T.
By Theorem II.9, we get
T = (TΓTΓa]∩(TΓaΓT ∪TΓTΓaΓTΓT]∩(aΓTΓT]
⊆ (a]∪ (TΓTΓa]∩(TΓaΓT ∪TΓTΓaΓTΓT]∩(aΓTΓT]
= Q(a).
Hence, T = Q(a).

(3)⇒(1) Assume that Q(a) = T for all a ∈ T. Let Q be an ordered quasi-Ternary Γ-ideal of T and let a ∈ Q. Then Q(a) = T, and so Q(a) ⊆ Q ⊆ T. Hence, T = Q. Therefore, T is quasi-simple.

Definition III.3: Let T be an ordered ternary Γ-semiring with a zero element, TΓTΓT ≠ {0} and |T| > 1. Then T is called ***0-quasi-simple*** if T has no nonzero proper ordered quasi-ternaryΓ-ideals.

Theorem III.4: Let T be an ordered ternary Γ-semiring with a zero element, TΓTΓT ≠ {0} and |T| > 1. Then T is 0-quasi-simple if and only if Q(a) = T for all a ∈ T \{0}.

Proof: Assume that T is 0-quasi-simple and let a ∈ T\{0}. Then Q(a) ≠ {0}.
Since T is 0-quasi-simple, we have Q(a) = T.
Conversely, assume that Q(a) = T for all a ∈ T \{0}. Let Q be a nonzero ordered quasi-Ternary Γ-ideal of T and a ∈ Q\{0}. Then Q(a) = T and Q(a) ⊆ Q ⊆ T. This implies that T = Q. Hence, T is 0-quasi-simple.

Definition III.5: An ordered quasi-Ternary Γ-ideal Q of an ordered ternary Γ-semiring T without a zero element is said to be a ***minimal ordered quasi-Ternary Γ-ideal*** of T if there is no an ordered quasi-Ternary Γ-ideal A of T such that A ⊂ Q. Equivalently, if for any ordered quasi-Ternary Γ-ideal A of T such that A ⊆ Q, we have A = Q.

Note III.6: We also define a ***minimal ordered left***, a ***minimal ordered lateral*** and a ***minimal ordered right ternary Γ- ideal*** of an ordered ternary Γ-semiring without a zero element in the same way of a minimal ordered quasi-Γ-ideal.

Theorem III.7: Let Q be an ordered quasi-ternary Γ-ideal of an ordered ternary Γ-semiring T without a zero element. Then Q is a minimal ordered quasi-ternary Γ-ideal of T if and only if it is the intersection of a minimal ordered left, a minimal ordered right and a minimal ordered lateral ternary Γ-ideal of T.

Proof: Assume that Q is a minimal ordered quasi-Ternary Γ-ideal of T.
Then (TΓTΓQ]∩(TΓQΓT ∪TΓTΓQΓTΓT]∩(QΓTΓT] ⊆ Q.
By Theorem II.5 (1), (TΓTΓQ], (QΓTΓT] and (TΓQΓT∪TΓTΓQΓTΓT] are an ordered left, an ordered right and an ordered lateral Ternary Γ-ideal of T, respectively, By Lemma II.4 (2), (TΓTΓQ]∩(TΓQΓT ∪TΓTΓQΓTΓT]∩(QΓTΓT] is an ordered quasi-Ternary Γ-ideal of T. Since Q is a minimal ordered quasi-Ternary Γ-ideal of T, we have (TΓTΓQ]∩(TΓQΓT ∪TΓTΓQΓTΓT]∩(QΓTΓT] = Q. We claim that (TΓTΓQ] is a minimal ordered left Ternary Γ-ideal of T. Let L be an ordered left Ternary Γ-ideal of T such that L ⊆ (TΓTΓQ]. Then (TΓTΓL] ⊆ (L] = L ⊆ (TΓTΓQ].
Thus (TΓTΓL]∩(TΓQΓT∪TΓTΓQΓTΓT]∩(QΓTΓT]
⊆ (TΓTΓQ]∩(TΓQΓT∪TΓTΓQΓTΓT]∩(QΓTΓT] = Q.

Since $(T\Gamma T\Gamma L]\cap(T\Gamma Q\Gamma T\cup T\Gamma T\Gamma Q\Gamma T\Gamma T]\cap(Q\Gamma T\Gamma T]$ is an ordered quasi-Ternary Γ-ideal of T and Q is a minimal ordered quasi-Ternary Γ-ideal of T,
we have $(T\Gamma T\Gamma L]\cap(T\Gamma Q\Gamma T\cup T\Gamma T\Gamma Q\Gamma T\Gamma T]\cap(Q\Gamma T\Gamma T] = Q$. Thus $Q \subseteq (T\Gamma T\Gamma L]$ and so $(T\Gamma T\Gamma Q] \subseteq (T\Gamma T\Gamma(T\Gamma T\Gamma L]] \subseteq (T\Gamma T\Gamma(L]] = (T\Gamma T\Gamma L] \subseteq L$. Hence, $L = (T\Gamma T\Gamma Q]$. Therefore, $(T\Gamma T\Gamma Q]$ is a minimal ordered left Ternary Γ-ideal of T. A similar proof holds for the other two case, $(Q\Gamma T\Gamma T]$ and $(T\Gamma Q\Gamma T \cup T\Gamma T\Gamma Q\Gamma T\Gamma T]$ are minimal ordered right and minimal ordered lateral Ternary Γ-ideal of T, respectively.

Conversely, let $Q = L\cap M\cap R$ where L, R and M are a minimal ordered left, a minimal ordered right and a minimal ordered lateral Ternary Γ-ideal of T, respectively. By Lemma II.4 (2), we have Q is an ordered quasi-Ternary Γ-ideal of T. Let A be an ordered quasi-Ternary Γ-ideal of T such that $A \subseteq Q$. By Theorem 2.5 (1), we have $(T\Gamma T\Gamma A]$, $(A\Gamma T\Gamma T]$ and $(T\Gamma A\Gamma T \cup T\Gamma T\Gamma A\Gamma T\Gamma T]$ are an ordered left, an ordered right and an ordered lateral Ternary Γ-ideal of T, respectively.

Now, $(T\Gamma T\Gamma A] \subseteq (T\Gamma T\Gamma Q] \subseteq (T\Gamma T\Gamma L) \subseteq (L] = L$. Since L is a minimal ordered left Ternary Γ-ideal of T, we have $(T\Gamma T\Gamma A] = L$. Similarly, $(A\Gamma T\Gamma T] = R$ and $(T\Gamma A\Gamma T \cup T\Gamma T\Gamma A\Gamma T\Gamma T] = M$. Since A is an ordered quasi-Ternary Γ-ideal of T, we have

$$Q = L\cap M \cap R = (T\Gamma T\Gamma A]\cap(T\Gamma A\Gamma T \cup T\Gamma T\Gamma A\Gamma T\Gamma T]\cap(A\Gamma T\Gamma T] \subseteq A.$$

This implies that A = Q. Hence, Q is a minimal ordered quasi-Ternary Γ-ideal of T.

Definition III.8: A nonzero ordered quasi-ternary Γ-ideal Q of an ordered ternary Γ-semiring T with a zero element is said to be a ***0-minimal ordered quasi-ternary Γ-ideal*** of T if there is no a nonzero ordered quasi-ternary Γ-ideal A of T such that $A \subset Q$. Equivalently, if for any nonzero ordered quasi-ternary Γ-ideal A of T such that $A \subseteq Q$, we have A = Q.

Note III.9: We also define a 0-minimal ordered left, a 0-minimal ordered lateral and a 0-minimal ordered right ternary Γ-ideal of an ordered ternary Γ-semiring with a zero element in the same way of a 0-minimal ordered quasi-ternary Γ-ideal.

Theorem III.10: Let T be an ordered ternary Γ-semiring with a zero element. Then the intersection of a 0-minimal ordered left, a 0-minimal ordered right and a 0-minimal ordered lateral ternary Γ-ideal of T is either {0} or a 0-minimal ordered quasi-ternary Γ-ideal of T.

Proof: Let $Q = L\cap M \cap R \neq \{0\}$ where L, R and M are a 0-minimal ordered left, a 0-minimal ordered right and a 0-minimal ordered lateral ternary Γ-ideal of T, respectively. By Lemma II.4 (2), we have Q is an ordered quasi-ternary Γ-ideal of T. Let A be a nonzero ordered quasi-ternary Γ-ideal of T such that $A \subseteq Q$. By Theorem II.5 (1), we have $(T\Gamma T\Gamma A]$, $(A\Gamma T\Gamma T]$ and $(T\Gamma A\Gamma T\cup T\Gamma T\Gamma A\Gamma T\Gamma T]$ are an ordered left, an ordered right and an ordered lateral ternary Γ-ideal of T, respectively. Thus we have the following two cases:

Case 1: $(T\Gamma T\Gamma A] = \{0\}$, $(A\Gamma T\Gamma T] = \{0\}$, or
$(T\Gamma A\Gamma T \cup T\Gamma T\Gamma A\Gamma T\Gamma T] = \{0\}$.
If $(T\Gamma T\Gamma A] = \{0\}$, then $(T\Gamma T\Gamma A] = \{0\} \subseteq A$. Thus A is a nonzero ordered left ternary Γ-ideal of T. Since $A \subseteq Q \subseteq L$ and L is a 0-minimal ordered left ternary Γ-ideal of T, we have A = L. This implies that A = Q. Similarly, if $(A\Gamma T\Gamma T] = \{0\}$ or $(T\Gamma A\Gamma T \cup T\Gamma T\Gamma A\Gamma T\Gamma T] = \{0\}$, then A = Q.

Case 2: $(T\Gamma T\Gamma A] \neq \{0\}$, $(A\Gamma T\Gamma T] \neq \{0\}$, and
$(T\Gamma A\Gamma T \cup T\Gamma T\Gamma A\Gamma T\Gamma T] \neq \{0\}$.
Now, $(T\Gamma T\Gamma A] \subseteq (T\Gamma T\Gamma Q] \subseteq (T\Gamma T\Gamma L) \subseteq (L] = L$. Since L is a 0-minimal ordered left ternary Γ-ideal of T, we have $(T\Gamma T\Gamma A] = L$.
Similarly, $(A\Gamma T\Gamma T] = R$ and $(T\Gamma A\Gamma T \cup T\Gamma T\Gamma A\Gamma T\Gamma T] = M$. Since A is an ordered quasi-ternary Γ-ideal of T, we have

$$Q = L\cap M \cap R = (T\Gamma T\Gamma A]\cap(T\Gamma A\Gamma T \cup T\Gamma T\Gamma A\Gamma T\Gamma T]\cap(A\Gamma T\Gamma T] \subseteq A.$$

This implies that A = Q. Hence, Q is a 0-minimal ordered quasi-ternary Γ-ideal of T.

Theorem III.11: Let Q be an ordered quasi-ternary Γ-ideal of an ordered ternary Γ-semiring T without a zero element. If Q is quasi-simple, then Q is a minimal ordered quasi-ternary Γ-ideal of T.

Proof: Assume that Q is quasi-simple and let A be an ordered quasi-ternary Γ-ideal of T such that $A \subseteq Q$. Now,

$$(Q\Gamma Q\Gamma A]\cap(Q\Gamma A\Gamma Q\cup Q\Gamma Q\Gamma A\Gamma Q\Gamma Q]\cap(A\Gamma Q\Gamma Q] \subseteq (T\Gamma T\Gamma A]\cap(T\Gamma A\Gamma T\cup T\Gamma T\Gamma A\Gamma T\Gamma T]\cap(A\Gamma T\Gamma T]\subseteq A$$

and $(A]\cap Q \subseteq (A] = A$. Thus A is an ordered quasi-ternary Γ-ideal of Q. Since Q is quasi-simple, we have A = Q. Hence, Q is a minimal ordered quasi-ternary Γ-ideal of T.

Theorem III.12: Let Q be a nonzero ordered quasi-ternary Γ-ideal of an ordered ternary Γ-semiring T with a zero element. If Q is 0-quasi-simple, then Q is a 0-minimal ordered quasi-ternary Γ-ideal of T.

Proof: Assume that Q is 0-quasi-simple and let A be a nonzero ordered quasi ternary Γ-ideal of T such that $A \subseteq Q$. Now,

$$(Q\Gamma Q\Gamma A]\cap(Q\Gamma A\Gamma Q\cup Q\Gamma Q\Gamma A\Gamma Q\Gamma Q]\cap(A\Gamma Q\Gamma Q] \subseteq (T\Gamma T\Gamma A]\cap(T\Gamma A\Gamma T\cup T\Gamma T\Gamma A\Gamma T\Gamma T]\cap(A\Gamma T\Gamma T] \subseteq A$$

and $(A]\cap Q \subseteq (A] = A$.
Thus A is a nonzero ordered quasi-ternary Γ-ideal of Q. Since Q is 0-quasi-simple, we have A = Q. Hence, Q is a 0-minimal ordered quasi-ternary Γ-ideal of T.

Theorem III.13: Let T be an ordered ternary Γ-semiring without a zero element having proper ordered quasi-ternary Γ-ideals. Then every proper ordered quasi-ternary Γ-ideal of T is minimal if and only if the intersection of any two distinct proper ordered quasi-ternary Γ-ideals is empty.

Proof: Let Q_1 and Q_2 be two distinct proper ordered quasi-ternary Γ-ideals of T. By assumption, we have that Q_1 and Q_2 are minimal. If $Q_1 \cap Q_2 \neq \emptyset$, then by Theorem II.7, $Q_1\cap Q_2$ is an ordered quasi-ternary Γ-ideal of T. Since $Q_1\cap Q_2 \subseteq Q_1$ and Q_1 is minimal, we have $Q_1\cap Q_2 = Q_1$. Since $Q_1\cap Q_2 \subseteq Q_2$ and Q_2 is minimal, we have $Q_1 = Q_1 \cap Q_2 = Q_2$. That is a contradiction. Hence, $Q_1 \cap Q_2 = \emptyset$.

Conversely, let Q be a proper ordered quasi-ternary Γ-ideal of T and let A be an ordered quasi-ternary Γ-ideal of T such that $A \subseteq Q$. Then A is a proper ordered quasi ternary Γ-ideal of T. If $A \neq Q$, then by assumption, $A = A \cap Q = \emptyset$. That is a contradiction. Hence, A = Q. Therefore, Q is a minimal ordered quasi-ternary Γ-ideal of T.

Theorem III.14:. Let T be an ordered ternary Γ-semiring with a zero element having nonzero proper ordered quasi-ternary Γ-ideals. Then every nonzero proper ordered quasi-ternary Γ-ideal of T is 0-minimal if and only if the intersection of any two distinct nonzero proper ordered quasi-ternary Γ-ideals is {0}.

Proof: Let Q_1 and Q_2 be two distinct proper ordered quasi-ternary Γ-ideals of T. By assumption, we have that Q_1 and Q_2 are minimal. If $Q_1 \cap Q_2 \neq \emptyset$, then by Theorem II.7, $Q_1 \cap Q_2$ is an ordered quasi-ternary Γ-ideal of T. Since $Q_1 \cap Q_2 \subseteq Q_1$ and Q_1 is 0-minimal, we have $Q_1 \cap Q_2 = Q_1$. Since $Q_1 \cap Q_2 \subseteq Q_2$ and Q_2 is 0-minimal, we have $Q_1 = Q_1 \cap Q_2 = Q_2$. That is a contradiction. Hence, $Q_1 \cap Q_2 = \emptyset$.

Conversely, let Q be a proper ordered quasi-ternary Γ-ideal of T and let A be a nonzero ordered quasi-ternary Γ-ideal of T such that A ⊆ Q. Then A is a proper ordered quasi ternary Γ-ideal of T. If A ≠ Q, then by assumption, A = A ∩ Q = {0}. That is a contradiction. Hence, A = Q. Therefore, Q is a 0-minimal ordered quasi-ternary Γ-ideal of T.

IV. Maximality Of Ordered Quasi-Ternary Γ-Ideals In Ordered Ternary Γ-Semirings

In this section, we characterize the relationship between the maximality of ordered quasi-ternary Γ-ideals and the union $\mathfrak{U}$ of all proper ordered quasi-ternary Γ-ideals in ordered ternary Γ-semiring without a zero element and the union $\mathfrak{U}_0$ of all nonzero proper ordered quasi-ternary Γ-ideals in ordered ternary Γ-semirings with a zero element.

Definition IV.1: A proper ordered quasi-ternary Γ-ideal Q of an ordered ternary Γ-semiring T is said to be a ***maximal ordered quasi-ternary Γ-ideal*** of T if there is no a proper ordered quasi-ternary Γ-ideal A of T such that Q ⊂ A. Equivalently, if for any proper ordered quasi-ternary Γ-ideal A of T such that Q ⊆ A, we have A = Q. Equivalently, if for any ordered quasi-ternary Γ-ideal A of T such that Q ⊂ A, we have A = T.

Theorem IV.2: Let Q be a proper ordered quasi-ternary Γ-ideal of an ordered ternary Γ-semiring T. If either
(1) T \Q = {a} for some a ∈ T or
(2) T \Q ⊆ (TΓTΓb]∩(TΓbΓT ∪TΓTΓbΓTΓT]∩(bΓTΓT] for all b ∈ T \Q,
then Q is a maximal ordered quasi-ternary Γ-ideal of T.

Proof: Let A be an ordered quasi-ternary Γ-ideal of T such that Q ⊂ A.

Case 1:
T \Q = {a} for some a ∈ T. Since Q ⊂ A, we have ∅ ≠ A\Q ⊆ T \Q = {a}. Thus A\Q = {a}.
Hence, A = Q∪(A\Q) = Q∪{a} = Q∪(T \Q) = T.

Case 2:
T \Q ⊆ (TΓTΓb]∩(TΓbΓT ∪TΓTΓbΓTΓT]∩(bΓTΓT] for all b ∈ T \Q. Let b ∈ A\Q ⊆ T \Q because A\Q ≠ ∅. Thus
T \Q ⊆ (TΓTΓb]∩(TΓbΓT ∪TΓTΓbΓTΓT]∩(bΓTΓT]
⊆ (TΓTΓA]∩(TΓAΓT ∪TΓTΓAΓTΓT]∩(AΓTΓT] ⊆ A.

Hence, T = Q∪(T\Q) ⊆ Q∪A = A. This implies that A = T. Therefore, Q is a maximal ordered quasi-ternary Γ-ideal of T.

Theorem IV.3: If Q is a maximal ordered quasi-ternary Γ-ideal of an ordered ternary Γ-semiring T and Q∪Q(a) is an ordered quasi-ternary Γ-ideal of T for all a ∈ T \Q, then either
(1) T\Q ⊆ (a] and $a\alpha a\beta a$ ∈ Q for some a ∈ T\Q, α, β ∈ Γ and (TΓTΓb]∩(TΓbΓT∪TΓTΓbΓTΓT]∩ (bΓTΓT] ⊆ Q for all b ∈ T \Q or
(2) T \Q ⊆ Q(a) for all a ∈ T \Q.

Proof: Assume that Q is a maximal ordered quasi-ternary Γ-ideal of an ordered ternary Γ-semiring T and Q∪Q(a) is an ordered quasi-ternary Γ-ideal of T for all a ∈ T \Q. Then we consider the following two cases:

Case 1: (TΓTΓa]∩(TΓaΓT ∪TΓTΓaΓTΓT]∩(aΓTΓT] ⊆ Q for some a ∈ T \Q.
Then $a\alpha a\beta a$ ∈ (TΓTΓa]∩(TΓaΓT ∪TΓTΓaΓTΓT]∩(aΓTΓT] ⊆ Q, so $a\alpha a\beta a$ ∈ Q.
By Theorem 2.9, we have
Q∪(a]=(Q∪((TΓTΓa]∩(TΓaΓT∪TΓTΓaΓTΓT]∩(aΓTΓT]))∪(a]
=Q∪(((TΓTΓa]∩(TΓaΓT ∪TΓTΓaΓTΓT]∩(aΓTΓT])∪(a])
= Q∪Q(a).
Thus Q∪(a] is an ordered quasi-ternary Γ-ideal of T. Since a ∈ T\Q, we have Q ⊂ Q∪(a]. Since Q is a maximal ordered quasi-ternary Γ-ideal of T, we have Q∪(a] = T. Thus T \Q ⊆ (a]. Next, we let b ∈ T \Q. Then b ≤ a. Thus
(TΓTΓb]∩(TΓbΓT ∪TΓTΓbΓTΓT]∩(bΓTΓT]
⊆ (TΓTΓa]∩(TΓaΓT ∪TΓTΓaΓTΓT]∩(aΓTΓT]
⊆ Q.

Case 2: (TΓTΓa]∩(TΓaΓT ∪TΓTΓaΓTΓT]∩(aΓTΓT] ⊈ Q for all a ∈ T \Q. Let a ∈ T \Q.

Then Q ⊂ Q∪Q(a). Since Q∪Q(a) is an ordered quasi-ternary Γ-ideal of T and Q is maximal, we have Q∪Q(a) = T. Hence, T \Q ⊆ Q(a).

Lemma IV.4: Let T be an ordered ternary Γ-semiring without a zero element. Then T = $\mathfrak{U}$ if and only if Q(a) ≠ T for all a ∈ T.

Proof: Assume that T = $\mathfrak{U}$ and let a ∈ T. Then a ∈ $\mathfrak{U}$, so a ∈ Q for some proper ordered quasi-ternary Γ-ideal Q of T. Hence, Q(a) ⊆ Q ≠ T, that is Q(a) ≠ T.
Conversely, assume that Q(a) ≠ T for all a ∈ T. Then Q(a) ⊆ U for all a ∈ T, so a ∈$\mathfrak{U}$ for all a ∈ T. Hence, T = $\mathfrak{U}$.

Theorem IV.5: Let T be an ordered ternary Γ-semiring without a zero element. Then one and only one of the following four conditions is satisfied:
(1) $\mathfrak{U}$ is not an ordered quasi-ternary Γ-ideal of T.
(2) Q(a) ≠ T for all a ∈ T.
(3) There exists a ∈ T such that Q(a) = T, (a]⊈(TΓTΓa]∩(TΓaΓT∪TΓTΓaΓTΓT]∩(aΓTΓT], and $a\alpha a\beta a$ ∈ $\mathfrak{U}$, T is not quasi-simple, T\$\mathfrak{U}$ = {x ∈ T | Q(x) = T}, and $\mathfrak{U}$ is the unique maximal ordered quasi-ternary Γ-ideal of T.
(4) T \$\mathfrak{U}$ ⊆ Q(a) for all a ∈ T \$\mathfrak{U}$, T is not quasi-simple, T \$\mathfrak{U}$ = {x ∈ T | Q(x) = T}, and $\mathfrak{U}$ is the unique maximal ordered quasi-ternary Γ-ideal of T.

Proof: Assume that $\mathfrak{U}$ is an ordered quasi-ternary Γ-ideal of T. We consider the following two cases:

Case 1: $\mathfrak{U}$ = T. By Lemma 4.4, the condition (2) holds.

Case 2: $\mathfrak{U} \neq$ T. Then T is not quasi-simple. We claim that $\mathfrak{U}$ is the unique maximal ordered quasi-ternary Γ-ideal of T. Let Q be an ordered quasi-ternary Γ-ideal of T such that $\mathfrak{U} \subset$ Q.

If Q ≠ T, then Q ⊆ $\mathfrak{U}$. That is a contradiction. Thus Q = T, so $\mathfrak{U}$ is a maximal ordered quasi-ternary Γ-ideal of T. Next, assume that A is a maximal ordered quasi-ternary Γ-ideal of T. Then A ≠ T, so A ⊆ $\mathfrak{U}$ ⊂ T. Since A is maximal, we have A = $\mathfrak{U}$. Therefore, $\mathfrak{U}$ is the unique maximal ordered quasi-ternary Γ-ideal of T. Since $\mathfrak{U}$ ≠ T, we have Q(a) = T for all $a \in T \setminus \mathfrak{U}$ and Q(a) ≠ T for all $a \in \mathfrak{U}$. Thus $T \setminus \mathfrak{U} = \{x \in T \mid Q(x) = T\}$ and so $\mathfrak{U} \cup Q(x)$ = T is an ordered quasi-ternary Γ-ideal of T for all $x \in T \setminus U$. By Theorem 4.3, we have the following two cases:

(i) $T\setminus U \subseteq (a]$ and $a\alpha a\beta a \in \mathfrak{U}$ for some $a \in T\setminus\mathfrak{U}$, and $(T\Gamma T\Gamma b] \cap (T\Gamma b\Gamma T \cup T\Gamma T\Gamma b\Gamma T\Gamma T] \cap (b\Gamma T\Gamma T] \subseteq \mathfrak{U}$ for all $b \in T \setminus \mathfrak{U}$ or

(ii) $T \setminus \mathfrak{U} \subseteq Q(a)$ for all $a \in T \setminus \mathfrak{U}$.

Assume (i) holds. Then T = Q(a).

If $(a] \subseteq (T\Gamma T\Gamma a] \cap (T\Gamma a\Gamma T \cup T\Gamma T\Gamma a\Gamma T\Gamma T] \cap (a\Gamma T\Gamma T]$, then by Theorem 2.9, we have

$$T = Q(a)$$
$$=(a] \cup ((T\Gamma T\Gamma a] \cap (T\Gamma a\Gamma T \cup T\Gamma T\Gamma a\Gamma T\Gamma T] \cap (a\Gamma T\Gamma T])$$
$$= (T\Gamma T\Gamma a] \cap (T\Gamma a\Gamma T \cup T\Gamma T\Gamma a\Gamma T\Gamma T] \cap (a\Gamma T\Gamma T] \subseteq \mathfrak{U}.$$

Thus $\mathfrak{U}$ = T. That is a contradiction.

Hence, $(a] \not\subseteq (T\Gamma T\Gamma a] \cap (T\Gamma a\Gamma T \cup T\Gamma T\Gamma a\Gamma T\Gamma T] \cap (a\Gamma T\Gamma T]$, so the condition (3) holds.

Assume (ii) holds. Then the condition (4) holds.

Note IV.6: For an ordered ternary Γ-semiring T with a zero element, the union of all nonzero proper ordered quasi-ternary Γ-ideals of T is denoted by $\mathfrak{U}_0$.

Lemma IV.7: Let T be an ordered ternary Γ-semiring with a zero element. Then T = $\mathfrak{U}_0$ if and only if Q(a) ≠ T for all $a \in$ T.

Proof: The proof is almost similar to the proof of Lemma IV.4.

Theorem IV.8: Let T be an ordered ternary Γ-semiring with a zero element. Then one and only one of the following four conditions is satisfied:

(1) $\mathfrak{U}_0$ is not an ordered quasi-ternary Γ-ideal of T.

(2) Q(a) ≠ T for all $a \in$ T.

(3) There exists $a \in$ T such that Q(a) = T, $(a] \not\subseteq (T\Gamma T\Gamma a] \cap (T\Gamma a\Gamma T \cup T\Gamma T\Gamma a\Gamma T\Gamma T] \cap (a\Gamma T\Gamma T]$, and $a\alpha a\beta a \in \mathfrak{U}_0$, T is not 0-quasi-simple, $T\setminus\mathfrak{U}_0 = \{x \in T \mid Q(x) = T\}$, and $\mathfrak{U}_0$ is the unique maximal ordered quasi-ternary Γ-ideal of T.

(4) $T\setminus\mathfrak{U}_0 \subseteq Q(a)$ for all $a \in T \setminus \mathfrak{U}_0$, T is not 0-quasi-simple, $T \setminus \mathfrak{U}_0 = \{x \in T \mid Q(x) = T\}$, and $\mathfrak{U}_0$ is the unique maximal ordered quasi-ternary Γ-ideal of T.

Proof: The proof is almost similar to the proof of Theorem IV.5.

CONCLUSIONS

In this paper mainly we start the study of ordered quasi-ternary Γ-ideals, in po-ternary Γ-semirings. We characterize minimality and maximality of ordered quasi ternary Γ-ideals in ternary Γ-semirings.

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