Minimality and Maximalitity of Ordered Quasi-Ternary $\Gamma$-Ideals in Ordered Ternary $\Gamma$-Semirings

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Abstract — Our aim in this paper is to develop a body of results on the minimality and maximality of ordered quasi-$\Gamma$-ideals in ordered ternary $\Gamma$-semirings, that can be used like the more classical results on unordered structures.

Index Terms — ordered ternary $\Gamma$-semiring, ordered quasi-$\Gamma$-ideal, minimality and maximality.

I. INTRODUCTION

The literature of ternary algebraic system was introduced by Lehmer [18] in 1932. He investigated certain ternary algebraic systems called triplexes which turn out to be ternary groups. The notion of ternary semigroups was known to Banach. He showed by an example that a ternary semigroup does not necessarily reduce to an ordinary semigroup. We can see that any semigroup can be reduced to a ternary semigroup. The study of ordered ternary semigroups began about 2000 by several authors, for example, Lampan [15], Chinram [8], and Akram and Yaqoob [1]. The theory of different types of ideals in (ordered) semigroups and in (ordered) ternary semigroups was studied by several researches such as: In 1965, Sioson [25] studied ideal theory in ternary semigroups. He also introduced the notion of regular ternary semigroups and characterized them by using the notion of quasi-ideals. In 1995, Dixit and Dewan [11] studied the properties of quasi-ideals and bi-ideals in ternary semigroups. In 1998, the concept and notion of ordered quasi-ideals in ordered semigroups was introduced by Kehayopulu [17]. In 2000, Cao and Xu [4] characterized minimal and maximal left ideals in ordered semigroups, and gave some characterizations of minimal and maximal left ideals in ordered semigroups. In 2002, Arslanov and Kehayopulu [2] gave some characterizations of minimal and maximal ideals in ordered semigroups. In 2004, Lampan and Siripitukdet [16] characterized (0-)minimal and maximal ordered left ideals in ordered $\Gamma$-semigroups, and gave some characterizations of (0-)minimal and maximal ordered left ideals in ordered $\Gamma$-semigroups. In 2007, Lampan [13] characterized (0-)minimal and maximal lateral ideals in ternary semigroups. In 2008, Lampan [14] characterized (0-)minimal and maximal ordered quasi-ideals in ordered semigroups, and gave some characterizations of (0-)minimal and maximal ordered quasi-ideals in ordered semigroups.


Before going to prove the main results we need the following definitions that we use later.
Definition I.1 [27]: Let $T$ and $\Gamma$ be two additive commutative semigroups. $T$ is said to be a ternary $\Gamma$-semiring if there exist a mapping from $T \times T \times T \times T \times T \times T$ to $T$ which maps $(x, y, z, a, b, c) \rightarrow [x, a, x, b, c]_\Gamma$ satisfying the conditions:

i) $[(a+b)c]d = [ac + bc]d = [a+bc]d = [ab + cd] = [abc]d$

ii) $[(a+b)c]d = [ac + bc]d = [a+bc]d$

iii) $[(a+b)c]d = [abc]d = [abcd]$ for all $a, b, c, d \in T$ and $a, b, c, d \in \Gamma$. Then $T$ is a ternary semiring.

Note I.2 [27]: Let $(T, \Gamma, +, \cdot)$ be a ternary $\Gamma$-semiring. For nonempty subsets $A_1, A_2$ and $A_3$ of $T$, let $[A_1 \Gamma A_2 \Gamma A_3] = \{a \in A_1 \cdot \beta \in A_2 \cdot a \cdot \beta \in A_3 \cdot \beta \in \Gamma \}$. For any $t \in T$, let $[t \Gamma A_1 \Gamma A_2] = \{t \cdot \beta \in A_1 \cdot \beta \in A_2 \cdot \beta \in \Gamma \}$. The other cases can be defined analogously.

Note I.3 [27]: Let $T$ be a ternary semiring. If $A, B$ are two subsets of $T$, we shall denote the set $A + B = \{a + b : a \in A, b \in B\}$ and $2A = \{a + a : a \in A\}$.

Definition I.4 [27]: A ternary $\Gamma$-semiring $T$ is called an ordered ternary $\Gamma$-semiring if there is a partial order $\leq$ on $T$ such that $x < y$ implies that (i) $a + c \leq b + c$ and $a + c < b + c$ (ii) $[a + b + c + d] < [a + b + c + d]$. For all $a, b, c, d \in T$ and $a, b, c, d \in \Gamma$.

Note I.5 [27]: For the convenience we write $x, a, x, b, x$ instead of $[x, a, x, b, x]$. 

Notation I.6 [27]: Let $T$ be PO-ternary $\Gamma$-semiring and $S$ be a nonempty subset of $T$. If $H$ is a nonempty subset of $S$, we denote $\{s \in S : s \leq h \}$ for some $h \in H$ by $[H]_S$.

Notation I.7 [27]: Let $T$ be PO-ternary $\Gamma$-semiring and $S$ be a nonempty subset of $T$. If $H$ is a nonempty subset of $S$, we denote $\{s \in S : h \leq s \}$ for some $h \in H$ by $[H]_S$.

Notation I.8 [27]: $[H]_r$ and $[H]_r$ are simply denoted by $[H]$ and $[H]$ respectively.

Definition I.9 [27]: Let $T$ be PO-ternary $\Gamma$-semiring. A nonempty subset $S$ is said to be a PO-ternary $\Gamma$-sub semi ring of $T$ if

i) $S$ is an additive semi group of $T$,

ii) $a \in S \subseteq S$ for all $a, b, c, d \in S$, $a, b, c, d \in \Gamma$.

iii) $T \subseteq S$, $a, c \leq S$, $a, c \leq \Gamma$. 

Note I.10 [27]: A nonempty subset $S$ of a po-ternary $\Gamma$-semiring $T$ is a po-ternary $\Gamma$-sub semi ring of $T$ if and only if $S \subseteq S$, $S \subseteq S$, $S \subseteq S$, $S \subseteq S$.


Definition I.12: An element $z$ of an ordered ternary $\Gamma$-semiring $T$ is said to be a zero element if (1) $z + z = z$ and (2) $z = z$ for all $x, y, z \in T$, $a, b, c \in \Gamma$ and if $z \in T$ is a zero element, it is denoted by 0.

Definition I.13 [27]: A nonempty subset $A$ of a po-ternary $\Gamma$-semiring $T$ is said to be left PO-ternary $\Gamma$-ideal of $T$ if

1) $a, b \in A$ implies $a + b \in A$.

2) $a, c \in T$, $a, b \in A$, $a, b \in \Gamma$ implies $bca \in A$.

3) $t \in T$, $a \in A$, $t \leq a \Rightarrow a \in A$.

Note I.14 [27]: A nonempty subset $A$ of a po-ternary $\Gamma$-semiring $T$ is a left PO-ternary $\Gamma$-ideal if and only if $A$ is additive subsemigroup of $T$, $A \subseteq A$ and $[A] \subseteq A$.

Definition I.15 [27]: A nonempty subset $A$ of a po-ternary $\Gamma$-semiring $T$ is said to be a lateral PO-ternary $\Gamma$-ideal of $T$ if

1) $a, b \in A$ implies $a + b \in A$.

2) $a, b \in T$, $a, b \in A$, $a, b \in \Gamma$ implies $bca \in A$.

3) $t \in T$, $a \in A$, $t \leq a \Rightarrow a \in A$.

Note I.16 [27]: A nonempty subset $A$ of a PO-ternary $\Gamma$-semiring $T$ is a lateral PO-ternary $\Gamma$-ideal of $T$ if and only if $A$ is additive sub semi group of $T$, $A \subseteq A$ and $[A] \subseteq A$.

Definition I.17 [27]: A nonempty subset $A$ of a PO-ternary $\Gamma$-semiring $T$ is a right PO-ternary $\Gamma$-ideal of $T$ if

1) $a, b \in A$ implies $a + b \in A$.

2) $a, b \in T$, $a, b \in A$, $a, b \in \Gamma$ implies $bca \in A$.

3) $t \in T$, $a \in A$, $t \leq a \Rightarrow a \in A$.

Note I.18 [27]: A nonempty subset $A$ of a PO-ternary $\Gamma$-semiring $T$ is a right PO-ternary $\Gamma$-ideal of $T$ if and only if $A$ is additive sub semi group of $T$, $A \subseteq A$ and $[A] \subseteq A$.

Definition I.19 [27]: A nonempty subset $A$ of a PO-ternary $\Gamma$-semiring $T$ is a two sided PO-ternary $\Gamma$-ideal of $T$ if

1) $a, b \in A$ implies $a + b \in A$.

2) $a, b \in T$, $a, b \in A$, $a, b \in \Gamma$ implies $bca \in A$.

3) $t \in T$, $a \in A$, $t \leq a \Rightarrow a \in A$.

Note I.20 [27]: A nonempty subset $A$ of a PO-ternary $\Gamma$-semiring $T$ is a two sided PO-ternary $\Gamma$-ideal of $T$ if and only if it is both a left PO-ternary $\Gamma$-ideal and a right PO-ternary $\Gamma$-ideal of $T$.

Definition I.21 [27]: A nonempty subset $A$ of a PO-ternary $\Gamma$-semiring $T$ is said to be PO-ternary $\Gamma$-ideal of $T$ if

1) $a, b \in A$ implies $a + b \in A$.

2) $a, b \in T$, $a, b \in A$, $a, b \in \Gamma$ implies $bca \in A$.

3) $t \in T$, $a \in A$, $t \leq a \Rightarrow a \in A$.

Note I.22 [27]: A nonempty subset $A$ of a PO-ternary $\Gamma$-semiring $T$ is an PO-ternary $\Gamma$-ideal if and only if it is left PO-ternary $\Gamma$-ideal, lateral PO-ternary $\Gamma$-ideal and right PO-ternary $\Gamma$-ideal of $T$.

II. ORDERED QUASI-TERNARY $\Gamma$-IDEALS AND ORDERED BINARY $\Gamma$-IDEALS
Definition II.1: An additive sub semi group $Q$ of an ordered ternary $Γ$-semiring $T$ is said to be an ordered quasi-Ternary $Γ$-ideal of $T$ if

1. $[TQT]∩[TQT]∩[QT] ⊆ Q$,
2. $[TQT]∩[TQT]∩[QT] ⊆ Q$, and
3. $[Q] ⊆ Q$.

Note II.2: We can easily prove that $\emptyset$ is the smallest ordered quasi-Ternary $Γ$-ideal of an ordered ternary $Γ$-semiring $T$ with a zero element and it is called a zero ordered quasi-Ternary $Γ$-ideal of $T$. Moreover, $0 ∈ Q$ for all ordered quasi-Ternary $Γ$-ideal $Q$ of $T$.

Definition II.3: An ordered ternary $Γ$-sub semi ring $B$ of an ordered ternary $Γ$-semiring $T$ is said to be an ordered bi-Ternary $Γ$-ideal of $T$ if

1. $BITBTBITB ⊆ B$, and
2. $[B] ⊆ B$.

Lemma II.4: Let $T$ be an ordered ternary $Γ$-semi ring. Then the following statements hold.

1. Every ordered left, ordered lateral and ordered right ternary $Γ$-ideal of $T$ is an ordered quasi-Ternary $Γ$-ideal of $T$.
2. The intersection of an ordered left, an ordered lateral and an ordered right $Γ$-ideal of $T$ is an ordered quasi-Ternary $Γ$-ideal of $T$.
3. Every ordered quasi-Ternary $Γ$-ideal of $T$ is an ordered bi-Ternary $Γ$-ideal of $T$.

Proof: (1) Let $L, R$, and $M$ be an ordered left, an ordered right and an ordered lateral Ternary $Γ$-ideal of $T$, respectively. (1) We see that $[L] = L, [R] = R$ and $[M] = M$.

Thus $[TTL]∩[TTL]∩[TTL] ⊆ [L] = L,$

$[TTT]∪[TTT]∪[TTT] ⊆ [R] = R$, and

$[TTTM]∩[TTTM]∩[TTTM] ⊆ [M] = M$.

Hence, $L, R$, and $M$ are ordered quasi-Ternary $Γ$-ideals of $T$.

(2) Suppose that $Q = L ∩ M ∩ R$ and let $l ∈ L, m ∈ M$ and $r ∈ R, α, β ∈ Γ$.

Then $ramβl ∈ RML ⊆ [TTL]∩[TTL]∩[TTL] ⊆ [L] = L,$

$[TT]∪[TT]∪[TT] ⊆ [R] = R$, and

$[TTTM]∩[TTTM]∩[TTTM] ⊆ [M] = M$.

Hence, $Q$ is an ordered quasi-Ternary $Γ$-ideal of $T$.

(3) Let $B$ be an ordered quasi-Ternary $Γ$-ideal of $T$. Then $BBBTBB ⊆ [TTL]∩[TTL]∩[TTL]$.

We have $BBBTBB ⊆ [TTT]∩[TTL]∩[TTL]$.

Then $B$ is an ordered quasi-Ternary $Γ$-ideal of $T$. Hence, $B$ is an ordered quasi-Ternary $Γ$-ideal of $T$.

Theorem II.5: Let $A$ be a nonempty subset of an ordered ternary $Γ$-semiring $T$. Then the following statements hold.

1. $[TTT]$, $[ATT]$ and $[TTA]∩[TTA]∩[TTA]$ are an ordered left, an ordered right and an ordered lateral Ternary $Γ$-ideals of $T$, respectively.

(2) $[TTA], [ATT] ∪ [A]$ and $[TAT]$ are an ordered left, an ordered right and an ordered lateral Ternary $Γ$-ideals of $T$ containing $A$, respectively.

Proof: (1) Suppose that $s, t ∈ [TTT]$. Then there exist $x, y, z, a, b, c, d ∈ T, c, d ∈ A$ such that $s = x, y, z, a, b, c, d ∈ T$ and $s + t = x, y, z, a, b, c, d ∈ T$. Since $T$ is a PO-ternary $Γ$-semiring and $[TTT]$ is a left PO-ternary $Γ$-ideal of $T$.

We have $s + t = x, y, z, a, b, c, d ∈ T$ and hence $s + t ∈ [TTT]$. Similarly we can show that $s + t ∈ [TTT]$. Therefore $[TTT]$ is additive sub semi groups of $T$.

Since $A ≠ \emptyset$, we have $[TTTA] ≠ \emptyset, [ATT] ≠ \emptyset$ and $[TTA] ≠ \emptyset$.

We see that $[TTT] = ([TTT]∪[ATT])∪[TTA]$. Thus $T = [TTT]∪[ATT]$.

Hence, $[TTTA] = [ATT]∪[TTA]$, and $[TTTA] = [ATT]∪[TTA]$.

Theorem II.6: If $Q$ is an ordered quasi-$Γ$-ideal of an ordered ternary $Γ$-semiring $T$, then it is the intersection of an ordered left, an ordered right and an ordered lateral ternary $Γ$-ideals of $T$, respectively.

Proof: Assume that $Q$ is an ordered quasi-Ternary $Γ$-ideal of $T$ and let $L = [TTL]∪[TT]∪[TTT]$ and $R = [TTT]∪[TT]∪[TTT]$.

By Theorem II.2, we have $L, R$ and $M$ are an ordered left, an ordered right and an ordered lateral Ternary $Γ$-ideals of $T$ containing $Q$, respectively. Thus $Q ⊆ L = \emptyset$.

Since $Q$ is an ordered quasi-Ternary $Γ$-ideal of $T$, we have $L = \emptyset$.

Hence, $Q = L = \emptyset$.

Theorem II.7: Let $T$ be an ordered ternary $Γ$-semiring. Then the intersection of arbitrary nonempty family of ordered quasi-Ternary $Γ$-ideals of $T$ is either empty or an ordered quasi-Ternary $Γ$-ideal of $T$.

Proof: Let $\{Q_i \mid i \in I\}$ be a nonempty family of ordered quasi-$Γ$-ideals of $T$ and let $Q = \bigcap_{i \in I} Q_i ≠ \emptyset$. We claim that $Q$ is an ordered quasi-Ternary $Γ$-ideal of $T$. Since $Q_i$ is an ordered quasi-Ternary $Γ$-ideal of $T$ for all $i \in I$, we have
(TTTTQ)∩(TTQΓTTTQTqTTT)∩(QTTTT) 
≤ (TTTTQ)∩(TTQΓTTTQΓTTT)∩(QTTTT) 
≤ Q, for all i ∈ I.
Thus (TTTTQ)∩(TTQΓTTTQΓTTT)∩(QTTTT) 
≤ \bigcap_{i=1}^{n} Q = Q and 

\[ Q = \bigcap_{i=1}^{n} Q_i \leq \bigcap_{i=1}^{n} Q = Q. \]

Hence, Q is an ordered quasi-Ternary Γ-ideal of T.

**Definition II.8:** Let A be an additive sub semi group of an ordered ternary Γ-semiring T. The intersection of all ordered quasi-ternary Γ-ideals of T containing A is called the **ordered quasi-Ternary Γ-ideal** of T generated by A and is denoted by Q(A). Moreover, Q(A) is the smallest ordered quasi-Ternary Γ-ideal of T containing A. If A = \{a\}, we also write Q(a) as Q(a).

**Theorem II.9:** Let A be an additive sub semi group of an ordered ternary Γ-semiring T. Then 

\[ Q(A) = \{A\} \cup (TTTTQ)\cap(TTQΓTTTQΓTTT)\cap(AQTTT) \]

In particular, 

\[ Q(a) = \{a\} \cup (TTTTQ)\cap(TTQΓTTTQΓTTT)\cap(aQTTT) \]

for all a ∈ T.

**Proof:** By Theorem II.5 (2), we have (AQTΓTTT) and (AQTΓTTT ∪ TTQΓTTT) are an ordered left, an ordered right and an ordered lateral ternary Γ-ideals of T containing A, respectively. By Lemma II.4 (2), we have (AQTΓTTT) ∩ (AQTΓTTT ∪ TTQΓTTT) is an ordered quasi-ternary Γ-ideal of T containing A. Thus \[ Q(A) ∈ (AQTΓTTT) ∩ (AQTΓTTT ∪ TTQΓTTT) = (AQTΓTTT) \]

By the proof of Theorem II.6, we have 

\[ \cup (AQTΓTTT) ∩ (AQTΓTTT ∪ TTQΓTTT) = (AQTΓTTT) \]

Since T is 0-quasi-simple, we have Q(a) = T.

**Note:** We also define a **minimal ordered left**, a **minimal ordered lateral** and a **minimal ordered right ternary Γ-ideal** of an ordered ternary Γ-semiring without a zero element in the same way of a minimal ordered quasi-Γ-ideal.

**III. MINIMALITY OF ORDERED QUASI-TERNARY Γ-IDEALS IN ORDERED TERNARY Γ-SEMIRINGS**

In this section, we characterize the relationship between the minimality of ordered quasi-ternary Γ-ideals and a quasi-simple and a 0-quasi-simple ordered ternary Γ-semirings.

**Definition III.1:** Let T be an ordered ternary Γ-semiring without a zero element. Then T is said to be **quasi-simple** if T has no proper ordered quasi-ternary Γ-ideals.

**Theorem III.2:** Let T be an ordered ternary Γ-semiring without a zero element. Then the following statements are equivalent.

(1) T is quasi-simple.

(2) (TTTTQ)∩(TTQΓTTTQΓTTT)∩(QTTTT) = T for all a ∈ T.

(3) Q(a) = T for all a ∈ T.

**Proof:** (1)⇒(2) Assume that T is quasi-simple and let a ∈ T. By Theorem II.5 (1), we have (TTTTQ)∩(aQTTT) and (TTQΓTTTQΓTTT) are an ordered left, an ordered right and an ordered lateral ternary Γ-ideals of T, respectively. By Lemma II.4 (2), we have (TTTTQ)∩(TTQΓTTTQΓTTT)∩(aQTTT) is an ordered quasi-Ternary Γ-ideal of T. Since T is quasi-simple, we have (TTTTQ)∩(TTQΓTTTQΓTTT)∩(aQTTT) = T.

(2)⇒(3) Assume that (TTTTQ)∩(TTQΓTTTQΓTTT)∩(aQTTT) = T for all a ∈ T. Let a ∈ T. Then (TTTTQ)∩(TTQΓTTTQΓTTT)∩(aQTTT) = T. By Theorem II.9, we get 

\[ T = (TTTTQ)∩(TTQΓTTTQΓTTT)∩(aQTTT) \]

Hence, T = Q(a).

**Note:** We also define a **minimal ordered left**, a **minimal ordered lateral** and a **minimal ordered right ternary Γ-ideal** of an ordered ternary Γ-semiring without a zero element in the same way of a minimal ordered quasi-Γ-ideal.

**Theorem III.7:** Let Q be an ordered quasi-ternary Γ-ideal of an ordered ternary Γ-semiring T without a zero element. Then Q is a minimal ordered quasi-ternary Γ-ideal of T if and only if it is the intersection of a minimal ordered left, a minimal ordered right and a minimal ordered lateral ternary Γ-ideal of T.

**Proof:** Assume that Q is a minimal ordered quasi-ternary Γ-ideal of T. Then (TTTTQ)∩(TTQΓTTTQΓTTT)∩(QTTTT) ⊆ Q. By Theorem II.5 (1), (TTTTQ) and (QTTTT) are an ordered left, an ordered right and an ordered lateral ternary Γ-ideal of T, respectively. By Lemma II.4 (2), (TTTTQ)∩(TTQΓTTTQΓTTT)∩(QTTTT) is an ordered quasi-ternary Γ-ideal of T. Since Q is a minimal ordered quasi-ternary Γ-ideal of T, we have (TTTTQ)∩(TTQΓTTTQΓTTT)∩(QTTTT) = Q. We claim that (TTTTQ) is a minimal ordered left ternary Γ-ideal of T. Let L be an ordered left ternary Γ-ideal of T such that L ⊆ (TTTTQ). Then (TTTTQ) ⊆ (L) = L ⊆ (TTTTQ).

Thus (TTTTQ)∩(TTQΓTTTQΓTTT)∩(QTTTT) \[ \subseteq (TTTTQ)∩(TTQΓTTTQΓTTT)∩(QTTTT) = Q. \]
Since $\langle \{\{T\}TT|\{TT\}TQ\}TT\{TT\}T\rangle \cap \{\{Q\}T\}TTT$ is an ordered quasi-Ternary $\Gamma$-ideal of $L$ and $M$ is a minimal ordered quasi-Ternary $\Gamma$-ideal of $T$, we have $\langle \{\{T\}TT|\{TT\}TQ\}TT\{TT\}T\rangle \cap \{\{Q\}T\}TTT = Q$. Thus $Q \subseteq \langle \{\{T\}TT|\{TT\}TQ\}TT\{TT\}T\rangle \subseteq \{\{TT\}T\}TTT \subseteq L$. Hence, $L = \{\{TT\}T\}TTT$. Therefore, $\{\{TT\}T\}TTT$ is a minimal ordered left Ternary $\Gamma$-ideal of $T$. A similar proof holds for the other two cases, $\{\{TT\}T\}TTT$ and $\{\{TT\}TQ\}TT\{TT\}T$ are minimal ordered right and minimal ordered lateral Ternary $\Gamma$-ideal of $T$, respectively.

Conversely, let $Q = L \cap M \cap R$ where $L$, $R$ and $M$ are a minimal ordered left, a minimal ordered right and a minimal ordered lateral Ternary $\Gamma$-ideal of $T$, respectively. By Lemma II.4 (2), we have $Q$ is an ordered quasi-Ternary $\Gamma$-ideal of $T$. Let $A$ be an ordered quasi-Ternary $\Gamma$-ideal of $T$ such that $A \subseteq Q$. By Theorem 2.5 (1), we have $\{\{TT\}T\}TTQ \subseteq \{\{TT\}T\}TTT \subseteq \{\{TT\}T\}TTT \subseteq L$. Hence, $L = \{\{TT\}T\}TTT$. Similarly, $\{\{TT\}T\}TTT \subseteq \{\{TT\}T\}TTT \subseteq \{\{TT\}T\}TTT \subseteq M$. Since $A$ is an ordered quasi-Ternary $\Gamma$-ideal of $T$, we have $Q = \langle \{\{TT\}T\}TTT \rangle \cap \{\{TT\}T\}TTT \subseteq A$. This implies that $A = Q$. Hence, $Q$ is a minimal ordered quasi-Ternary $\Gamma$-ideal of $T$.

**Definition III.8:** A nonzero ordered quasi-Ternary $\Gamma$-ideal $Q$ of an ordered ternary $\Gamma$-semiring $T$ with a zero element is said to be a **0-minimal ordered quasi-Ternary $\Gamma$-ideal** of $T$ if there is no a nonzero ordered quasi-Ternary $\Gamma$-ideal $A$ of $T$ such that $A \subseteq Q$. Equivalently, if for any nonzero ordered quasi-Ternary $\Gamma$-ideal $A$ of $T$ such that $A \subseteq Q$, we have $A = Q$.

**Note III.9:** We also define a 0-minimal ordered left, a 0-minimal ordered right and a 0-minimal ordered lateral Ternary $\Gamma$-ideal of $T$ is either $\{0\}$ or a 0-minimal ordered quasi-Ternary $\Gamma$-ideal of $T$.

**Theorem III.10:** Let $T$ be an ordered ternary $\Gamma$-semiring with a zero element. Then the intersection of a 0-minimal ordered left, a 0-minimal ordered right and a 0-minimal ordered lateral Ternary $\Gamma$-ideal of $T$ is either $\{0\}$ or a 0-minimal ordered quasi-Ternary $\Gamma$-ideal of $T$.

**Proof:** Let $Q = L \cap M \cap R \neq \{0\}$ where $L$, $R$ and $M$ are a 0-minimal ordered left, a 0-minimal ordered right and a 0-minimal ordered lateral Ternary $\Gamma$-ideal of $T$, respectively. By Lemma II.4 (2), we have $Q$ is an ordered quasi-Ternary $\Gamma$-ideal of $T$. Let $A$ be a nonzero ordered quasi-Ternary $\Gamma$-ideal of $T$ such that $A \subseteq Q$. By Theorem II.5 (1), we have $\{\{TT\}T\}TTQ \subseteq \{\{TT\}T\}TTT \subseteq \{\{TT\}T\}TTT \subseteq L$. Hence, $L = \{\{TT\}T\}TTT$. Similarly, $\{\{TT\}T\}TTT \subseteq \{\{TT\}T\}TTT \subseteq \{\{TT\}T\}TTT \subseteq M$. Since $A$ is an ordered quasi-Ternary $\Gamma$-ideal of $T$, we have $Q = \{\{TT\}T\}TTT \subseteq A$. This implies that $A = Q$. Hence, $Q$ is a 0-minimal ordered quasi-Ternary $\Gamma$-ideal of $T$.

**Theorem III.11:** Let $Q$ be an ordered quasi-Ternary $\Gamma$-ideal of an ordered ternary $\Gamma$-semiring $T$ without a zero element. If $Q$ is quasi-simple, then $Q$ is a 0-minimal ordered quasi-Ternary $\Gamma$-ideal of $T$.

**Proof:** Assume that $Q$ is quasi-simple and let $A$ be an ordered quasi-Ternary $\Gamma$-ideal of $T$ such that $A \subseteq Q$. Now,

$$\{\{TT\}T\}TTQ \subseteq \{\{TT\}T\}TTT \subseteq \{\{TT\}T\}TTT \subseteq M.$$ 

This implies that $A = Q$. Hence, $Q$ is a 0-minimal ordered quasi-Ternary $\Gamma$-ideal of $T$.

**Theorem III.12:** Let $Q$ be a nonzero ordered quasi-Ternary $\Gamma$-ideal of an ordered ternary $\Gamma$-semiring $T$ with a zero element. If $Q$ is quasi-simple, then $Q$ is a 0-minimal ordered quasi-Ternary $\Gamma$-ideal of $T$.

**Proof:** Assume that $Q$ is 0-quasi-simple and let $A$ be a nonzero ordered quasi ternary $\Gamma$-ideal of $T$ such that $A \subseteq Q$. Now,

$$\{\{TT\}T\}TTQ \subseteq \{\{TT\}T\}TTT \subseteq \{\{TT\}T\}TTT \subseteq M.$$ 

Thus $A$ is a 0-minimal ordered quasi-Ternary $\Gamma$-ideal of $Q$. Since $Q$ is 0-quasi-simple, we have $A = Q$. Hence, $Q$ is a 0-minimal ordered quasi-Ternary $\Gamma$-ideal of $T$.

**Theorem III.13:** Let $T$ be an ordered ternary $\Gamma$-semiring without a zero element having proper ordered quasi-Ternary $\Gamma$-ideals. Then every proper ordered quasi-Ternary $\Gamma$-ideal of $T$ is minimal if and only if the intersection of any two distinct proper ordered quasi-Ternary $\Gamma$-ideals is empty.

**Proof:** Let $Q_1$ and $Q_2$ be two distinct proper ordered quasi-Ternary $\Gamma$-ideals of $T$. By assumption, we have that $Q_1 \cap Q_2$ is minimal. If $Q_1 \cap Q_2 \neq \emptyset$, then by Theorem II.7, $Q_1 \cap Q_2$ is an ordered quasi-Ternary $\Gamma$-ideal of $T$. Since $Q_1 \cap Q_2 \subseteq Q_1$ and $Q_2$ is minimal, we have $Q_1 \cap Q_2 = Q_1$. Since $Q_1 \cap Q_2 \subseteq Q_2$ and $Q_2$ is minimal, we have $Q_2 = Q_1 \cap Q_2 = Q_2$. That is a contradiction. Hence, $Q_1 \cap Q_2 = \emptyset$.

Conversely, let $Q$ be a proper ordered quasi-Ternary $\Gamma$-ideal of $T$ and let $A$ be an ordered quasi-Ternary $\Gamma$-ideal of $T$ such that $A \subseteq Q$. Then $A$ is a proper ordered quasi ternary $\Gamma$-ideal of $T$. If $A \neq Q$, then by assumption, $A = A \cap Q = \emptyset$. That is a
contradiction. Hence, \( A = Q \). Therefore, \( Q \) is a minimal ordered quasi-ternary \( \Gamma \)-ideal of \( T \).

Theorem III.14: Let \( T \) be an ordered ternary \( \Gamma \)-semiring with a zero element having nonzero proper ordered quasi-ternary \( \Gamma \)-ideals. Then every nonzero proper ordered quasi-ternary \( \Gamma \)-ideal of \( T \) is 0-minimal if and only if the intersection of any two distinct nonzero proper ordered quasi-ternary \( \Gamma \)-ideals is \( \{0\} \).

Proof: Let \( Q_1 \) and \( Q_2 \) be two distinct proper ordered quasi-ternary \( \Gamma \)-ideals of \( T \). By assumption, we have that \( Q_1 \cap Q_2 \) is a minimal ideal of \( T \). Since \( Q_1 \cap Q_2 \) is a minimal ideal of \( T \), we have \( Q_1 \cap Q_2 = \{0\} \). Since \( Q_1 \cap Q_2 \neq \{0\} \), we have \( Q_1 \cap Q_2 = Q_2 \). That is a contradiction. Hence, \( Q_1 \cap Q_2 = \{0\} \).

Conversely, let \( Q \) be a proper ordered quasi-ternary \( \Gamma \)-ideal of \( T \) and let \( A \) be a nonzero ordered quasi-ternary \( \Gamma \)-ideal of \( T \) such that \( A \subseteq Q \). Then \( A \) is a proper ordered quasi-ternary \( \Gamma \)-ideal of \( T \). If \( A \neq \{0\} \), then by assumption, \( A = A \cap Q = \{0\} \). That is a contradiction. Hence, \( A = Q \). Therefore, \( Q \) is a 0-minimal ordered quasi-ternary \( \Gamma \)-ideal of \( T \).

IV. Maximality of Ordered Quasi-Ternary \( \Gamma \)-Ideals in Ordered Ternary \( \Gamma \)-Semirings

In this section, we characterize the relationship between the maximality of ordered quasi-ternary \( \Gamma \)-ideals and the union \( \bigcup \) of all proper ordered quasi-ternary \( \Gamma \)-ideals in ordered ternary \( \Gamma \)-semiring without a zero element and the union \( \bigcup \) of all nonzero proper ordered quasi-ternary \( \Gamma \)-ideals in ordered ternary \( \Gamma \)-semirings with a zero element.

Definition IV.1: A proper ordered quasi-ternary \( \Gamma \)-ideal \( Q \) of an ordered ternary \( \Gamma \)-semiring \( T \) is said to be a maximal ordered quasi-ternary \( \Gamma \)-ideal of \( T \) if there is no proper ordered quasi-ternary \( \Gamma \)-ideal of \( T \) such that \( Q \subseteq A \). Equivalently, if for any proper ordered quasi-ternary \( \Gamma \)-ideal \( A \) of \( T \) such that \( Q \subseteq A \), we have \( A = Q \). Equivalently, if for any ordered quasi-ternary \( \Gamma \)-ideal \( A \) of \( T \) such that \( Q \subseteq A \), we have \( A = T \).

Theorem IV.2: Let \( Q \) be a proper ordered quasi-ternary \( \Gamma \)-ideal of an ordered ternary \( \Gamma \)-semiring \( T \). If either

1. \( T \setminus Q = \{a\} \) for some \( a \in T \) or
2. \( T \setminus Q \subseteq \{bTT\} \) \( \cup \{TT\} \) \( \cup \{bbTT\} \) \( \cup \{bTT\} \) for all \( b \in T \setminus Q \),

then \( Q \) is a maximal ordered quasi-ternary \( \Gamma \)-ideal of \( T \).

Proof: Let \( A \) be an ordered quasi-ternary \( \Gamma \)-ideal of \( T \) such that \( Q \subseteq A \).

Case 1: \( T \setminus Q = \{a\} \) for some \( a \in T \). Since \( Q \subseteq A \), we have \( A \setminus Q \subseteq T \setminus Q = \{a\} \). Therefore, \( A \setminus Q = \{a\} \).

Case 2: \( T \setminus Q \subseteq \{TTTT\} \cup \{TT\} \cup \{bTT\} \cup \{bbTT\} \) for all \( b \in T \setminus Q \). Let \( b \in A \setminus Q \subseteq T \setminus Q \) because \( A \setminus Q \neq \emptyset \).

Thus \( T \setminus Q \subseteq \{TTTT\} \cup \{TT\} \cup \{bTT\} \cup \{bbTT\} \) \( \subseteq \{TTTT\} \cup \{TT\} \cup \{bTT\} \cup \{bbTT\} \subseteq A \).

Hence, \( T = QU(T \setminus Q) \subseteq QUA = A \). This implies that \( A = T \). Therefore, \( Q \) is a maximal ordered quasi-ternary \( \Gamma \)-ideal of \( T \).

Theorem IV.3: If \( Q \) is a maximal ordered quasi-ternary \( \Gamma \)-ideal of an ordered ternary \( \Gamma \)-semiring \( T \) and \( QU(a) \) is an ordered quasi-ternary \( \Gamma \)-ideal of \( T \) for all \( a \in T \setminus Q \), then either

1. \( T \setminus Q \subseteq \{a\} \) and \( a \neq \emptyset \) \( \cup \{TTTT\} \cup \{TT\} \cup \{bTT\} \cup \{bbTT\} \subseteq Q \) for all \( b \in T \setminus Q \) or
2. \( T \setminus Q \subseteq QU(a) \) for all \( a \in T \setminus Q \).

Proof: Assume that \( Q \) is a maximal ordered quasi-ternary \( \Gamma \)-ideal of an ordered ternary \( \Gamma \)-semiring \( T \) and \( QU(a) \) is an ordered quasi-ternary \( \Gamma \)-ideal of \( T \) for all \( a \in T \setminus Q \). Then we consider the following two cases:

Case 1: \( \{TTTT\} \) \( \cup \{TT\} \cup \{bTT\} \cup \{bbTT\} \subseteq Q \) for some \( a \in T \setminus Q \).

Thus \( QU(a) = Q \) is an ordered quasi-ternary \( \Gamma \)-ideal of \( T \). Since \( a \in T \setminus Q \), we have \( Q \subseteq QU(a) \). Since \( Q \) is a maximal ordered quasi-ternary \( \Gamma \)-ideal of \( T \), we have \( QU(a) = T \). Thus \( T \setminus Q \subseteq \{a\} \). Next, we let \( b \in T \setminus Q \). Then \( b \leq a \). Thus \( \{TTTT\} \cup \{TT\} \cup \{bTT\} \cup \{bbTT\} \subseteq \{TTTT\} \cup \{TT\} \cup \{bTT\} \cup \{bbTT\} \subseteq Q \).

Case 2: \( \{TTTT\} \) \( \cup \{TT\} \cup \{bTT\} \cup \{bbTT\} \subseteq Q \) for all \( a \in T \setminus Q \).

Then \( Q \subseteq QU(a) \). Since \( QU(a) \) is an ordered quasi-ternary \( \Gamma \)-ideal of \( T \) and \( Q \) is maximal, we have \( QU(a) = T \). Hence, \( T \setminus Q \subseteq QU(a) \).

Lemma IV.4: Let \( T \) be an ordered ternary \( \Gamma \)-semiring without a zero element. Then \( T = U \) if and only if \( QU(a) \neq T \) for all \( a \in T \).

Proof: Assume that \( T = U \) and let \( a \in T \). Then \( a \in U \), so \( a \in Q \). However, \( a \in Q \) and \( Q \subseteq T \). Conversely, assume that \( Q(a) \neq T \) for all \( a \in T \). Then \( Q(a) \subseteq U \) for all \( a \in T \), so \( a \in U \) for all \( a \in T \). Hence, \( T = U \).

Theorem IV.5: Let \( T \) be an ordered ternary \( \Gamma \)-semiring without a zero element. Then one and only one of the following four conditions is satisfied:

1. \( U \) is not an ordered quasi-ternary \( \Gamma \)-ideal of \( T \).
2. \( Q(a) \neq T \) for all \( a \in T \).
3. There exists \( a \in T \) such that \( Q(a) = T \), \( a \in T \), \( a \notin Q \), and \( QU(a) \) is not quasi-simple, \( T \setminus a = \{x \in T \mid Q(x) = T\} \) and \( U \) is the unique maximal ordered quasi-ternary \( \Gamma \)-ideal of \( T \).
4. \( T \setminus U \subseteq QU(a) \) for all \( a \in T \setminus U \) and \( Q \) is not quasi-simple, \( T \setminus U \subseteq \{x \in T \mid Q(x) = T\} \) and \( U \) is the unique maximal ordered quasi-ternary \( \Gamma \)-ideal of \( T \).
Proof: Assume that $U$ is an ordered quasi-ternary $Γ$-ideal of $T$. We consider the following two cases: 

**Case 1:** $U = T$. By Lemma 4.4, the condition (2) holds.

**Case 2:** $U \neq T$. Then $T$ is not quasi-simple. We claim that $U$ is the unique maximal ordered quasi-ternary $Γ$-ideal of $T$. Let $Q$ be an ordered quasi-ternary $Γ$-ideal of $T$ such that $U \subseteq Q$. 

If $Q \neq T$, then $Q \subseteq U$. That is a contradiction. Thus $Q = T$, so $U$ is a maximal ordered quasi-ternary $Γ$-ideal of $T$. Next, assume that $A$ is a maximal ordered quasi-ternary $Γ$-ideal of $T$. Then $A \neq T$, so $A \subseteq U \subseteq T$. Since $A$ is maximal, we have $A = U$. Therefore, $U$ is the unique maximal ordered quasi-ternary $Γ$-ideal of $T$. Since $U \neq T$, we have $Q(a) = T$ for all $a \in T \setminus U$ and $Q(a) \neq T$ for all $a \in U$. Thus $T \setminus U = \{x \in T \mid Q(x) = T\}$ so $U \subseteq Q(x) = T$ is an ordered quasi-ternary $Γ$-ideal of $T$ for all $x \in T \setminus U$. By Theorem 4.3, we have the following two cases:

(i) $T \setminus U \subseteq \{a \setminus \} and $α(β)\setminus \{\alpha\setminus \}$ for some $a \in T \setminus U$, and $(TTT\alpha\setminus \{\alpha\setminus \})\cap(\alpha\setminus \}) \subseteq \{\alpha\setminus \}$ for all $\beta \in T \setminus U$ or

(ii) $T \setminus U \subseteq \{a \setminus \} for all $a \in T \setminus U$.

Assume (i) holds. Then $T = Q(a) = \{a\setminus \}(TTT\alpha\setminus \{\alpha\setminus \})\cap(\alpha\setminus \}) \subseteq \{\alpha\setminus \}$. Thus $U \subseteq T$. That is a contradiction.

Hence, $\{a \setminus \}(TTT\alpha\setminus \{\alpha\setminus \})\cap(\alpha\setminus \}) \subseteq U$, so the condition (3) holds.

Assume (ii) holds. Then the condition (4) holds.

**Note IV.6:** For an ordered ternary $Γ$-semiring $T$ with a zero element, the union of all nonzero proper ordered quasi-ternary $Γ$-ideals of $T$ is denoted by $U_0$.

**Lemma IV.7:** Let $T$ be an ordered ternary $Γ$-semiring with a zero element. Then $T = U_0$ if and only if $Q(a) \neq T$ for all $a \in T$.

**Proof:** The proof is almost similar to the proof of Lemma IV.4.

**Theorem IV.8:** Let $T$ be an ordered ternary $Γ$-semiring with a zero element. Then one and only one of the following four conditions is satisfied:

1. $U_0$ is not an ordered quasi-ternary $Γ$-ideal of $T$.
2. $Q(a) \neq T$ for all $a \in T$.
3. There exists $a \in T$ such that $Q(a) = T$.
4. $T \setminus U_0 \subseteq Q(a)$ for all $a \in T \setminus U_0$. $T$ is not $0$-quasi-simple, $T \setminus U_0 = \{x \in T \mid Q(x) = T\}$, and $U_0$ is the unique maximal ordered quasi-ternary $Γ$-ideal of $T$.

**Proof:** The proof is almost similar to the proof of Theorem IV.5.

**References**


[23] Sajani Lavanya, Madhusudhana Rao and Syam Julius Rajendra., on Quasi-Ternary $\Gamma$-Ideals and Bi-Ternary $\Gamma$-Ideals in Ternary $\Gamma$-Semirings accepted for publication in International Journal of Mathematics and Statistics Invention in the month October 2015.


